Problem 1 (10%)

A first-order system with a sinusoidal input can be generally written as the following equation:

$$\tau y' + y = A\sin\omega t$$

where τ is the system time constant, ω is the frequency and A is the input amplitude.

(a) Find the general solution of this equation. **(5%)**

(b) If $\tau = 1$, $A = 1$, $\omega = 1$ rad/s, $y(0) = -2$. Please find it's solution and draw the curve of $y(t)$. **(5%)**

Problem 2 (10%)

Given a mass-spring-damper system, with unknown values of K and C, an impulse function $r(t) = \delta(t)$ generates an output response as $y(t) = e^{-t} - e^{-2t}$.

Now if we are given another input function $r(t) = \sin t$, please find the corresponding output response.

Problem 3 (15%)

(a) Given a matrix A=$\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$, find the orthonormal matrix T that can produce the diagonal matrix D=$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ by $D = TAT^{-1}$, where λ_1 and λ_2 are eigenvalues and $\lambda_1 > \lambda_2$. **(5%)**

(b) Given the quadratic form $x^T Ax = Q$ where $x = [x_1, x_2]^T$, prove that $\lambda_1 y_1^2 + \lambda_2 y_2^2 = Q$ if $y = T^{-1}x$, where $y = [y_1, y_2]^T$. **(5%)**

(c) Identify the conic section $5x_1^2 - 2x_1x_2 + 5x_2^2 = 24$ and plot the graph of the conic section. **(5%)**

Problem 4 (15%)

(a) Please find a set of eigenfunctions $\{y_m\}$ $m = 1, 2, 3, \ldots$ that can satisfy the periodic Sturm-Liouville problem

$$y'' + \lambda y = 0,\ \ y(\pi) = y(-\pi),\ \ y'(\pi) = y'(-\pi).\ \ \textbf{(5\%)}$$

(b) Show that a function $f(x)$ can be represented by an orthogonal expansion

$f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$. Find the representation of a_m. **(5%)**

(c) What is the relationship between $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$ and the Fourier Series? **(5%)**

Problem 5 (15%)

Prove the following expressions:

(a) If $\mathbf{A}$ is a vector with constant magnitude, show that $\mathbf{A}$ and $d\mathbf{A}/dt$ are perpendicular provided $|d\mathbf{A}/dt| \neq 0$. **(5%)**

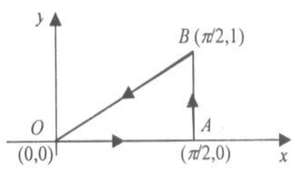
(b) $\nabla^2\left(\frac{1}{r}\right) = 0$, where r is the magnitude of the vector $\mathbf{r}$ which is defined as $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. **(5%)**

(c) $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$, where $\mathbf{A}$ is a vector. **(5%)**

Problem 6 (10%)

Evaluating the following integrations:

(a) $\oint_C (y - \sin x)dx + \cos x dy$, where C is the triangle shown in the following figure. **(5%)**



(b) $\iint_S \vec{r}\cdot\vec{n}\,dS$ = ? where S is a closed surface and $\vec{n}$ is the positive (outward drawn) normal to S. **(5%)**

Problem 7 (10%)

(a) Find the Laurent series of $f(z)=(z-2)\sin\left(\frac{1}{z+2}\right)$ about the indicated singularity z = -2. Name the singularity and give the region of convergence the series. **(5%)**

(b) Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent series valid for $|z|<1$. **(5%)**

Problem 8 (7%)

Find the residue of the complex function $f(z)=\frac{\cot z\coth z}{z^3}$ at $z=0$.

Problem 9 (8%)

Prove that $\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx=\pi\ln 2$.