Problem 1 (20%) Solve the following differential equations,

(a) $x\,\mathrm{d}y + \left(3y - 2x^2\right)\mathrm{d}x = 0$ (10%)

(b) $y''' - 3y'' = 2 - \cos x$ (10%)

Problem 2 (10%) Solve the following differential equation by Laplace Transform method,

$y'' + y = t^2 + 4\sin(2t)$ with initial conditions, $y(0) = 0$ and $y'(0) = 0$

Problem 3 (20%) Solve the following differential equation by the method of finding the eigenvalues and eigenvectors of matrix A,

$$Y' = \mathrm{A}\cdot Y + \begin{bmatrix} 3x \\ e^{-x} \end{bmatrix} \text{ where } \mathrm{A} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

Problem 4 (20%)

(a) Expand $f(x) = \sin x$, $0 < x < \pi$, in a Fourier cosine series. (10%)

(b) If $f(x)$ is an even function show that the Fourier transform of $f(x)$ is

$$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u)\cos(\omega u)\,du.$$ (5%)

(c) Show that $\int_0^\infty \frac{\cos \omega x}{\omega^2 + 1} d\omega = \frac{\pi}{2} e^{-x}$. (5%)

Problem 5 (15%)

(a) Solve the equation $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$ (5%) and find the particular solution for which

$z(x,0) = x^2, z(1,y) = \cos y$. (5%)

(b) Find a general solution for $t\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial u}{\partial x} = x^2$. (5%)

Problem 6 (15%)

(a) Prove $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$. (5%)

(b) For a complex variable, $z = x + iy$, evaluate the integration of $\int_{1+i}^{2+4i} z^2\,dz$ along the parabola $x = t, y = t^2$ where $1 \le t \le 2$. (5%)

(c) Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$. (5%)