Problem 1 (20%) Solve the following differential equations,

(a) $\frac{dy}{dx} = \frac{y \cdot x^2 \cdot ln(x)}{(y+3)^2}$ (10%)

(b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x \cdot e^{-x}$ (10%)

Problem 2 (15%) Solve the following differential equation by Laplace Transform method,

$$\frac{d^2y}{dt^2} + 9y = f(t) \quad \text{where} \quad f(t) = \begin{cases} 0, & 0 < t < \pi \\ 2, & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$$

with initial conditions: y(0) = 0 and y'(0) = 1

Problem 3 (15%) Given a matrix A, as shown below, find a 3×3 matrix M such that $\left(M^{-1} \cdot A \cdot M\right) = D$ is diagonal. Show the matrix M and the diagonal matrix D.

$$A = \begin{bmatrix} 3 & -1 & 5 \\ -1 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

Problem 4 (20%)

(a) Prove that $\lim_{M\to\infty}\int_{-\pi}^{\pi} f(x)\sin\left(M+\frac{1}{2}\right)x\,dx=0$, if $f(x)$ is piecewise continuous and M is an integer. **(5%)**

(b) Find the Fourier transform of $f(x)=\begin{cases}1 & |x|<a\\ 0 & |x|>a\end{cases}$. **(5%)**

(c) Use the result of (b), evaluate $\int_{-\infty}^{\infty}\frac{\sin ua\cos ux}{u}\,du$=? **(5%)**

(d) Evaluate $\int_0^{\infty}\frac{x^2}{(x^2+1)}\,dx$ by use of Parseval's identity. **(5%)**

Problem 5 (15%)

(a) Show that $y(x,t)=F(2x+5t)+G(2x-5t)$ is a general solution of $4\frac{\partial^2 y}{\partial t^2}=25\frac{\partial^2 y}{\partial x^2}$. **(5%)**

(b) Solve $\frac{\partial u}{\partial x}=4\frac{\partial u}{\partial y}$, $u(0,y)=8e^{-3y}$. **(5%)**

(c) Solve $\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2}$ for $0<x<3$ and $t>0$ subjected to the following boundary and initial conditions as $u(0,t)=u(3,t)=0$ and $u(x,0)=5\sin 4\pi x-3\sin 8\pi x+2\sin 10\pi x$. **(5%)**

Problem 6 (15%)

(a) If C is a simple closed curve bounding a region having $z = a$ as interior point, find $\oint_C\frac{dz}{(z-a)^n}$=? where $n = 1, 2, 3, 4, 5$ …. **(5%)**

(b) For the following two expressions, locate in the finite z plane all the singularities and name them the type of singularity. **(5%)**

(i) $\frac{\sin mz}{z^2+2z+2}, m\neq 0$. (ii) $e^{-1/(z-1)^2}$

(c) Evaluate $\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$=?. **(5%)**