Problem 1 (30%)

Solve the following differential equations,

(a) $y' + xy - 4x = 0$ (10%)

(b) $y'' + 4y' + 29y = 1$ with y(0)=0 and $y'(0) = 3$ (10%)

(c) $y'' - 4y' + 4y = x^3 e^{2x}$ with y(0)=0 and $y'(0) = 0$ (10%)

Problem 2 (10%)

Determine the rank of the following matrix as a function of k.

$$\begin{bmatrix} (k-2) & 2 & -1 \\ 2 & (k-2) & -1 \\ -1 & -1 & (2k-3) \end{bmatrix}$$

Problem 3 (10%)

Given 3 data points located in the xy coordinated systems are (0, -0.5), (1, 3.5), and (3, 15). Find the best straight line that fits (least squares) to the above data.

Problem 4 (10%)

(a) Expand $f(x) = x^2, 0 < x < 2\pi$ in a Fourier series if the period is 2π. (6%)

(b) Find $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = ?$ (4%)

Problem 5 (15%)

(a) If $f(x)$ is an even function, find the Fourier transform of $f(x)$, $F(\alpha)=?$ (8%)

(b) Based on (a), if $\int_0^\infty f(x)\cos\alpha x\,dx = \begin{cases} 1-\alpha & 0\le\alpha\le 1 \\ 0 & \alpha>1 \end{cases}$, find $f(x)=?$ (4%)

(c) Use (b), show that find $\int_0^\infty \frac{\sin^2 x}{x^2}dx = ?$. (3%)

Problem 6 (10%)

(a) Assuming both functions ϕ_1 and ϕ_2 are at least twice differentiable and c is any constant, prove that $u(x,t)=\phi_1(x-ct)+\phi_2(x+at)$ satisfies $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$. (6%)

(b) If the initial conditions are given as $\begin{cases} u(x,0)=f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x) \end{cases}$, for $-\infty \le x \le \infty$, find the solution for $u(x,t)$=? (4%)

Problem 7 (15%)

(a) If z is a complex variable, $z=x+iy$, evaluate $\oint_C \frac{5z^2-3z+2}{(z-1)^3}dz=?$ Here, C is any simple closed curve enclosing z = 1. (5%)

(b) Determine the residues of $\frac{1}{z(z+2)^2}$ at the poles z = 0, and z = -2. (5%)

(c) Find $\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)^2(x^2+2x+2)}=?$ (5%)