1. (10%)

Suppose that $\mathbf{K}$ is a square matrix with $\mathbf{K} = -\mathbf{K}^{T}$ and that $\mathbf{I}-\mathbf{K}$ is nonsingular; define

$$\mathbf{A} = (\mathbf{I}+\mathbf{K})(\mathbf{I}-\mathbf{K})^{-1}.$$

Show that $\mathbf{A}$ is an *orthogonal matrix*.

2. (10%)

Use the *Laplace transform* to solve the initial-value problem

$$x'' + 4x = \sin 3t\,,\quad x(0) = 0,\quad x'(0) = 0$$

3. (10%)

Solve the differential equation by variation of parameters

$$y'' - 2y' + y = \frac{e^{x}}{1+x^{2}}$$

4. (10%)

Find the eigenvalues and the associated eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} &y'' + \lambda y = 0, \quad (0 < x < L) \\ &y'(0) = 0 \qquad y(L) = 0 \end{aligned}$$

5. (10%)

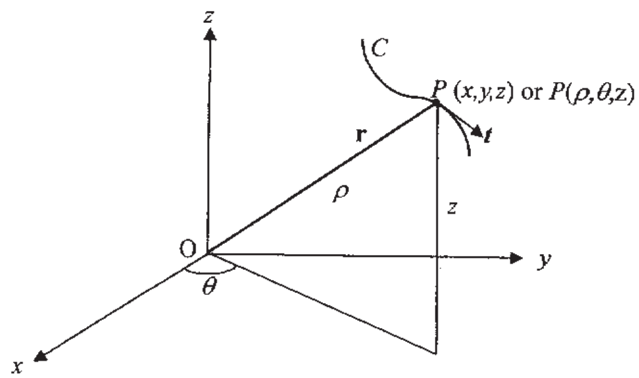
Use separation of variables to solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

6. (15%)

(a) If a scalar ϕ and a vector **A** are given as $\phi = x^2yz^3$ and $\mathbf{A} = xz\mathbf{i} - y2\mathbf{j} + 2x^2y\mathbf{k}$, respectively. Find: Gradient ϕ=? ($\nabla\phi$=?), Divergence **A** =? ($\nabla \cdot \mathbf{A}$=?) and Curl **A** =? ($\nabla \times \mathbf{A}$ =?) **(6%)**

(b) Referring to the following figure, a particle is located at point *P* on a curve *C*. If **r** is the vector joining the origin O and the point P. In the Cartesian coordinates, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let *s* be the arc length to (*x,y,z*) measured from the initial position, **dr**/ds = **t**, where **t** is a unit vector in the direction of the tangent to the curve and $ds = \sqrt{d\mathbf{r} \cdot d\mathbf{r}}$. Show that $(ds)^2 = (d\rho)^2 + \rho^2(d\theta)^2 + (dz)^2$ in the cylindrical coordinates (ρ, θ, z). **(5%)**



(c) On the curve *C* shown in the above figure, if *F*(*x,y,z*) is defined at any point. Show that $\frac{dF}{ds} = \nabla F \cdot \mathbf{t}$, where **t** is a unit vector defined in (b). **(4%)**

7. **(20%)**

(a) Find the Fourier series corresponding to the function

$$f(x)=\begin{cases}0 & -5<x<0\\ 3 & 0<x<5\end{cases}\quad \text{Period}=10 \qquad \textbf{(5\%)}$$

(b) Expand $f(x)=x, 0<x<2$, in a half range cosine series. **(5%)**

(c) Find the Fourier transform of $f(x)=\begin{cases}1 & |x|<5\\ 0 & |x|>5\end{cases}$ **(5%)**

(d) Evaluate the integration $\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}d\alpha=?$ **(5%)**

8. **(15%)**

Let z is a complex variable and is defined as $z=x+iy$.

(a) If $u=e^{-x}(x\sin y-y\cos y)$, find v such that the complex function $f(z)=u+iv$ is analytic. **(5%)**

(b) Evaluate $\int_C (z+2)e^{iz}dz$ along the parabola C defined by $\pi^2 y=x^2$ from (0,0) to (π, 1). **(5%)**

(c) Evaluate $\oint_C \frac{e^{zu}}{z^2(z^2+2z+2)}dz$ around the circle C with equation $|z|=5$. **(5%)**