1. (15%) Let $\mathbf{u}_1 = (1,1,1)^\top, \mathbf{u}_2 = (1,2,2)^\top, \mathbf{u}_3 = (2,3,4)^\top$.
   (a) (10%) Find the transition matrix corresponding to the change of basis from $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.
   (b) (5%) Find the coordinates of $(2,3,2)^\top$ with respect to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

2. (15%) Let A be a 2×2 matrix, and let L be the linear operator defined by

$$L(\mathbf{x}) = \mathtt{A}\mathbf{x}$$

   Show that
   (a) (10%) L maps R^2 onto the column space of A.
   (b) (5%) If A is nonsingular, then L maps R^2 onto R^2.

3. (10%) Solve the initial-value problem

$$x^2y'' - 5xy' + 10y = 0, \quad y(1) = 1,\ y'(1) = 0.$$

4. (10%) Find a general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix}, -\infty < t < \infty.$$

5. Let $\mathbf{v} = 2x^3\boldsymbol{i} + (y+z)^2\boldsymbol{j} + xyz\boldsymbol{k}, \boldsymbol{w} = (x-y)^2\boldsymbol{i} + 3z\boldsymbol{j} + 2yx\boldsymbol{k}$ (Assume the coordinate system to be right-handed whenever this is essential.)
   Find
   (a) grad (div $\mathbf{w}$) $\cdot$ $\mathbf{v}$ (10%)
   (b) div (curl($\mathbf{v} + \mathbf{w}$)) (10%)

6. Assume the external force to be sinusoidal, say, $\mathrm{P} = \mathrm{A}\rho\sin\omega t$. Show that

$$\frac{\mathrm{P}}{\rho} = \mathrm{A}\sin\omega t = \sum_{n=1}^{\infty} k_n(t) sin\frac{n\pi x}{L}$$

   Where $\mathrm{k}_n(t) = (2A/n\pi)(1 - cosn\pi)sin\omega t$; consequently $\mathrm{k}_n = 0(n\ even)$, and $\mathrm{k}_n = (2A/n\pi)sin\omega t(n\ odd)$. (10%)

7. Let $\mathrm{w} = \mathrm{f}(\mathrm{x},\mathrm{y},\mathrm{z})$, and let $\mathrm{z} = \mathrm{g}(\mathrm{x},\mathrm{y})$ represent a surface S in space. Then on S, the function becomes $\vec{w}(x,y) = f[x,y,g(x,y)]$
   Show that its partial derivatives are obtained from

$$\frac{\partial \vec{\mathrm{w}}}{\partial \mathrm{x}} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}, \qquad \frac{\partial \vec{\mathrm{w}}}{\partial \mathrm{y}} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}$$

   Apply this to $f = x^3 + y^3 + z^2$, $g = x^2 + y^2$ and check by substitution and direct differentiation. (20%)